PHYSICAL REVIEW E, VOLUME 65, 031310

Basin of attraction of a bounded self-organized critical state

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(Received 21 August 2001; published 5 March 2002)

The robustness of the self-organized critical (SOC) state observed in the motion of an annular plate rotating over a granular medium is studied in this paper. In particular, we investigate the effect of parameters to which the emergent SOC state may be sensitive, including the initialization scheme, driving velocity, and confining pressure. The results indicate that the critical state is not a universal attractor, but has a finite basin of attraction. Furthermore, this state is only one of the three observed, which compare well with subcritical, critical, and supercritical states. The results call into question the precise definition of the term ‘‘self-organized criticality,’’ an issue we address.

DOI: 10.1103/PhysRevE.65.031310 PACS number(s): 45.70.Cc, 05.65.+b, 91.30.−f

INTRODUCTION

In the theory of self-organized criticality (SOC), introduced by Bak, Tang, and Wiesenfeld (BTW) in the late 1980s [1], it is suggested that certain classes of spatially extended systems are inherently attracted to a critical state, thereby explaining the scale invariance observed in diverse natural systems. We have designed and implemented a mechanical apparatus that allows us to observe and record the stick-slip motion of an annular plate over a granular tapioca bed. The system is driven slowly by a motor via a torsion spring. In this way, the torque on the annular plate builds up until a frictional threshold is exceeded, at which point a slip event occurs.

We have previously introduced the background for this research, and have concluded that SOC is present in the system’s dynamics [2,3]. To understand the mechanical origin of the SOC state, we considered the second-order phase transition observed in the shear modulus of a compressed granular medium [4,5]. At this point, the coordination number (the average number of contacts per grain) has been observed to obey a first-order transition when free of gravity, and we suggested that this may change to a second-order transition under the influence of gravity. This transition point is consistent with long-range order spontaneously occurring along connected chains of particles (i.e., ‘‘stress chains’’) [6,7], even though grains normally interact only with neighboring grains. Though this transition point occurs only at a ‘‘critical’’ volume fraction [4], our system can automatically move to this volume fraction by dilating into the gap between the annular plate and the channel walls. The power-law distribution of events macroscopically observed by us is the system’s response to the slow driving. That this occurs under nonspecific conditions (i.e., the medium is randomized and leveled, and the annular plate is placed on top) suggests that this complexity is self-organized.

The focus of this paper is to establish any limits of the emergent SOC state with respect to variations in the system’s operating parameters. The results presented in this paper are typical of the set of experiments to which each result pertains. The effect of the following variables has been studied: initialization scheme; driving rate; and external confining pressure.

SYSTEM STATES

In general, three *states* or *modes of operation* may be distinguished. We denote these three states as low (L), medium (M), and high (H), according to the steady-state torque exerted by the torsion spring, shown in Fig. 1. The nature of the variation in the torque also changes between the states. For state L, the torque remains almost steady with only small fluctuations, whereas for states M and H, the torque can vary over a significant range. In state H, the device is also subject to recurrent large events.

The event size distributions for the three states are shown in Fig. 2. State L obeys a power-law distribution with a cutoff of $S_0 \sim 1°$, state M a power-law distribution $P(S) \sim S^{-\lambda}$, $\lambda \simeq 1.9$ with a cutoff of $S_0 \sim 20°$, and state H also obeys a power law but with a lognormal surplus of large events centered at approximately $34°$.

The event energy and duration distributions obey similar patterns though each are subject to oscillations that are due

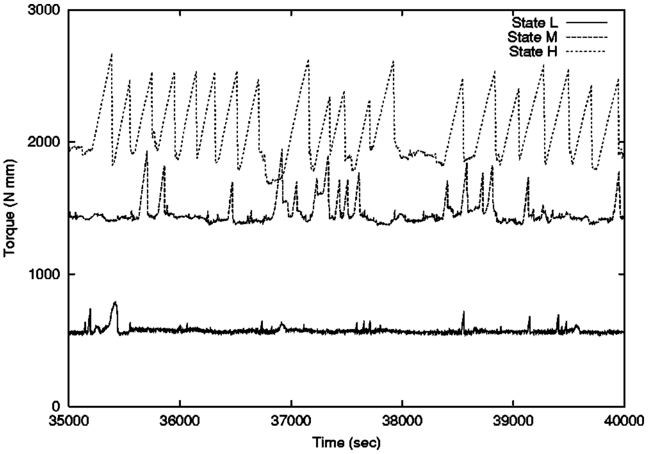

FIG. 1. Torque as a function of time for states L, M, and H. The three experiments clearly demonstrate qualitatively different stick-slip motion.

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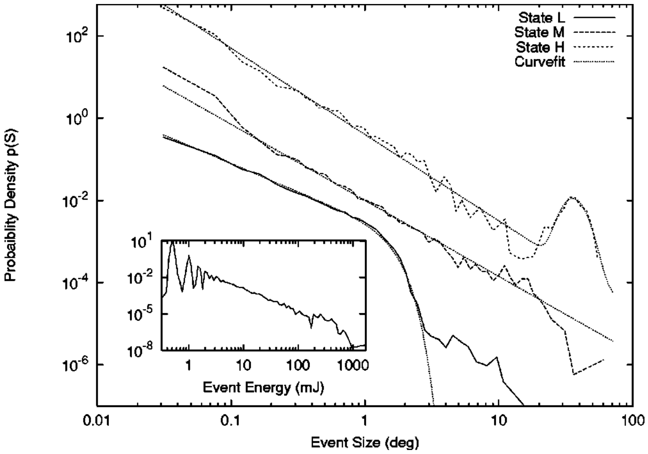

FIG. 2. Distribution of event sizes *S* for states *L*, *M*, and *H*, with curve fits to each. The curves are shifted vertically for clarity. The inset shows the event energy distribution for the state *M*. Oscillations at the left are due to the discrete nature of the measurement system.

to the discrete nature of the measurement system. The distribution of event energies for the state *M* is shown in the inset of Fig. 2.

The power-law exponents for these experiments are shown in Table I. Note that a power law is a valid description of the event duration distributions for state *L* distributions in only 50% of the relevant experiments, and so an exponent is not quoted for this data. However, the curves obtained can be generalized to a gamma distribution: $P(D) \sim D^{-\tau} e^{-bD^{\beta}}$. In addition, a $1/f^2$ power spectrum is observed over five orders of magnitude in all three states.

From the descriptions outlined above, we deduce that the three states *L*, *M*, and *H* correspond to subcritical, critical, and supercritical states, as observed for tectonic activity [8]. It would be interesting to compare the sequence of events for such activity to those obtained from these experiments. In addition, though the three states were initially categorized by their steady-state torque, it is now clear that other criteria can also be identified, specifically the form of the event distributions, and the character of the motion observed.

Thus the subcritical state is an easily identifiable state of the system with (a) a low torque value, (b) event distributions with a small cutoff, and (c) at best a questionable power law for event duration distributions. The supercritical state is also easily identifiable with (a) a high torque value, (b) recurrent large events, and (c) power-law event distributions with a Gaussian-type surplus of large events. These two states are also highly repeatable.

TABLE I. Power-law exponents for event distributions and the power spectra for each state.

State	Size λ	Energy α	Duration τ	Power spectrum
L	1.4±0.02	1.36±0.03	-	1.96±0.03
M	1.94±0.03	1.88±0.04	2.08±0.04	2.03±0.03
H	1.99±0.05	1.95±0.05	2.08±0.06	1.99±0.02

The critical state exhibits a much richer variety of behaviors and can often exhibit behavior that is, in one fashion or another, more characteristic of the other two states. In fact, when attempting to select an experiment that was "most representative" of the critical state, it emerged that no single experiment did so perfectly; all experiments deviated in some way from what we abstractly consider to be the "ideal" critical state experiment. Despite this, other features of each experiment ensured the correct classification.

INITIALIZATION SCHEME

We have found that the operating state of the system can be influenced by the choice of the initialization scheme, that is, the method by which the apparatus is prepared before any experiment. This was investigated as follows. The granular medium was first combed to its maximum depth (typically 25–30 mm) in order to randomize the configuration. If the top plate was now simply placed on top of the medium and the experiment run, the subcritical state was the predominant outcome. The critical state was induced by combing the medium as above, and then compacting and smoothing by manually rotating the top plate back and forth over the medium before starting the experiment. If the top plate was also forcibly embedded into the medium by (for example) 5–10 mm at this stage, the supercritical state would arise. Intuitively, we believe that these initialization schemes result in a medium that is, respectively, uncompressed, compressed and highly compressed.

Consider now the origin of the subcritical state. The initialization process, which leads to this state, results in a medium that would have a low volume fraction. As a result, the stress chains, which support the torque applied by the torsion spring, would be quite weak, and few in number [4]. As such the medium would be incapable of supporting a large torque, and the cutoff for the size of events would be considerably smaller than for the higher torque states. In addition, the lower shear modulus brought about by a weaker network of stress chains would enable easier fluidization of the medium. Hence a predominantly fluidized system is observed, with occasional bursts of activity when the medium enters a configuration where a higher torque is sustainable.

In the supercritical state, the medium is initially overcompacted, with a high volume fraction. Thus the network of force chains would be quite dense and strong [4], and the system would rarely enter a configuration where the maximum sustainable torque is low. The rupture of a single stress chain would have little effect, as particles in neighboring chains would be sufficiently supported to retain stability. Hence a predominantly solid system is observed, with occasional bursts of activity when the medium enters a configuration where a higher torque is not sustainable. We suspect that the rise time of the recurrent events is related to the fracture stress of individual particles; it may be that the jamming state entered can only be broken by the destruction of one or more key particles. This would naturally occur at a characteristic torque that would arise a characteristic time after the jamming state is entered. Indeed, we have generally observed that, after a number of experiments, a small quan-

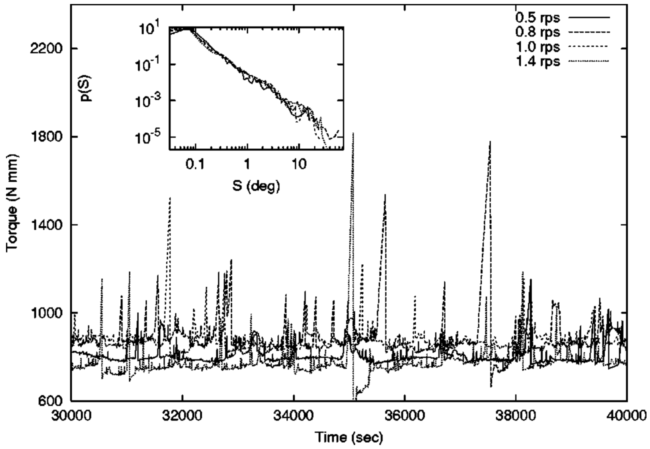

FIG. 3. Evolution of the torque for four experiments run at different driving speeds, from 0.6 to 1.45 mrps. The inset shows the probability distribution of event sizes for the four experiments.

tity of tapioca dust accumulates at the bottom of the channel. Despite the regular appearance of the data in Fig. 1, there appears to be no characteristic time interval between jamming events, as evidenced by the absence of individual Fourier components in the power spectra for these experiments

It is notable that the large events in state *H* are similar to ‘‘resetting’’ events observed in a number of sandpile experiments [9–11]. An excess of large events has also been recently observed in a numerical model of a compressed granular medium exhibiting shear [12]. This suggests that these systems are also driven to a supercritical state.

To summarize, if the system is initialized such that the dynamics can eject sufficient matter out of the channel and sufficiently compact the lower layers of the medium, the system will approach the critical state. If the initialization means that not enough matter can be ejected, the supercritical state is entered, and if there is not enough matter to give full contact between the plate and the medium, the subcritical state arises.

In the remaining sections, we investigate the robustness only of the critical state with respect to variation in operating conditions.

DRIVING RATE

The apparatus is driven by a variable speed motor via a torsion spring, resulting in an angular driving velocity of $0.2 \leqslant \omega \leqslant 32° \mathrm{s}^{-1}$. The torsion spring permits the system to store a certain amount of potential energy, which is partly released during a slip event. In this section, the behavior of the system is studied for four different driving rates: 0.61, 0.82, 1.0 and 1.45 $\times 10^{-3}$ revolutions per second (mrps) (these correspond to a driving motor voltage of 2.5, 3.0, 3.5, and 4.0 V respectively).

In Fig. 3 the steady state torque of the device is shown for each driving rate. It is clear that the torque is not significantly affected by the choice of the driving speed. The inset plots the distribution of event sizes for the same experiments. The power laws do not change significantly, an observation repeated with the event energy distribution. In contrast, however, the driving rate has a clear effect on the distribution of event durations, shown in Fig. 4; namely, a higher driving speed moves the lower end of the distribution to smaller time scales. By rescaling the data along the event duration axis by an amount approximately equal to the driving rate ($\omega^{0.9}$), the humped features at the lower end of the distribution can be made to coincide, as shown in the inset.

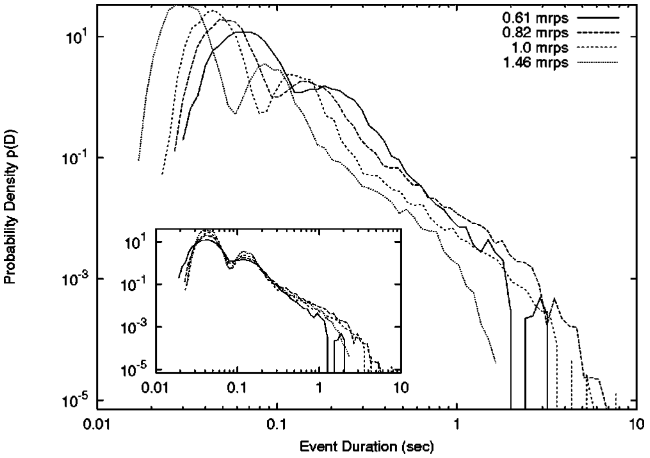

FIG. 4. Distribution of event durations for the four experiments run at different driving speeds, from 0.6 to 1.45 mrps. The inset shows the same data rescaled along the duration axis.

The origin of this effect lies in the definition of ‘‘an event.’’ An event is defined to start when the top plate exceeds the rotation speed of the driving, and to stop when the speed drops below the driving speed. Hence a faster driving rate will require that an event must occur more rapidly. Thus the distribution of events will be pushed to shorter durations with faster driving. As event size increases, the exact termination point of the event becomes logarithmically less significant and so large events are not subject to this effect.

In Fig. 5 the power spectra of the fluctuations in the torque are shown for the four experiments. The main graph shows the four curves overlaid on one another, and the inset

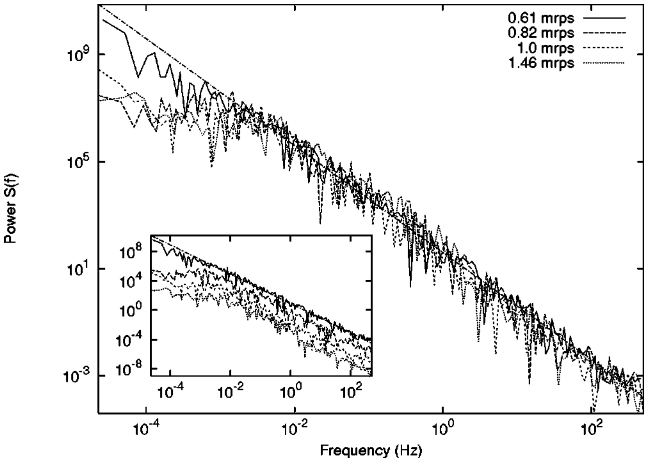

FIG. 5. Power spectra of experiments run at different driving speeds. Three experiments show a transition from $1/f^0$ to $1/f^2$ noise at approximately 3 mHz. The data are vertically offset for clarity in the inset.

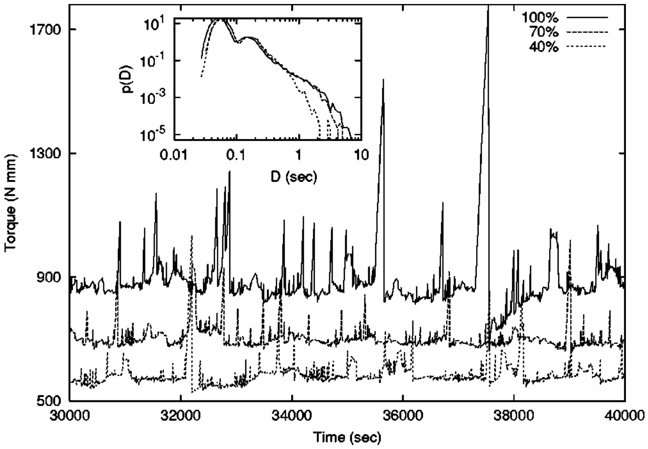

FIG. 6. Variation of the steady state torque with top plate load. The inset shows the variation of duration distribution.

shows them vertically offset. For all experiments, there is a transition to $1/f^2$ noise, which occurs at approximately 3 mHz. For the 0.6 mrps experiment, the spectrum below this point exhibits a $1/f^{1.5}$ spectrum, where the others exhibit white noise. Other experiments at the same driving rate do not exhibit this feature, instead reproducing the white noise spectrum of the three other curves. The origin of the $1/f^{1.5}$ spectrum in this experiment is unclear.

Overall, therefore, there is no change in the critical behavior that arises from increasing the rate at which the system is driven, apart from apparently decreasing the duration of events, an effect that is merely an artifact of the detection system.

CONFINING PRESSURE

The apparatus is equipped with a load-modifying assembly mounted over the top plate's rotation axis. By inserting a compression or extension spring into this assembly, it is possible to shift weight between the granular medium and the axle. In this manner, we investigate here the behavior observed when the pressure between the top plate and the granular medium is 100%, 70%, and 40% of the top plate's weight (approximately 10.8 N).

It is important to note, however, that the confining pressure will only affect initial conditions in the apparatus, as friction between the top plate and the axle upon which it is mounted prevents the plate from moving vertically once the torque exceeds approximately 100 N mm (well below the typical steady state torque, which is of the order 500–2000 N mm). Therefore, the top plate effectively provides a constant volume for the medium. However, as previously explained [2], the medium may dilate through the gap between the plate and the channel walls.

The steady state torque of the device for the three experiments is shown in Fig. 6. The torque is seen to reduce from an average of 860 N mm at 100%, through 700 N mm at 70%, to 570 N mm at 40%. This trend is generally observed throughout the experiments, though the variable nature of the critical state means there are exceptions. Recall that though the three states were initially identified by their steady state torque, it emerged that the nature of the event distributions together with the steady state torque and the type of motion observed (solid, fluid or both) established the true classification. In this section, the three experiments shown also have three levels of torque, but none of them exhibit the properties clearly exhibited by the subcritical or supercritical state, but they do exhibit properties consistent with the critical state. Table II illustrates the mean steady state torque for the three top plate loadings, averaged over several experiments. It is clear that the load applied effects a change.

TABLE II. Torque and power-law exponents for experiments of differing top plate loads, averaged over all relevant experiments.

Load	100%	70%	40%
Avg. torque	900 ± 60	730 ± 40	580 ± 20
Size λ	2.0 ± 0.15	2.0 ± 0.2	2.08 ± 0.05
Energy α	1.85 ± 0.15	1.80 ± 0.05	1.8 ± 0.2
Duration τ	2.1 ± 0.1	2.1 ± 0.1	3.4 ± 0.15

As with the variation with motor speed, the distributions of the event size and energy are unaffected by the change in top plate load. Once again, however, the duration distribution is affected. The inset to Fig. 6 demonstrates the duration distributions for the same set of experiments. The 100% and 70% experiments exhibit the same power-law decay with $\tau = 2.1$, whereas $\tau = 3.4$ applies for the 40% experiments. Table II also illustrates the mean exponents for these experiments. This variation is attributed to the system approaching the transition between the critical and subcritical states. However, as the experiment does not exhibit the small cutoff value for the event size and energy, and still obeys a power law for event duration, it is categorized as a critical experiment. Other experiments conducted at 40% of the top plate load exhibit either the steep duration power laws here or the shallower power law more consistent with the 100% and 70% experiments.

Thus, it is observed that the steady state torque at which the device operates, is reduced as the pressure is lowered. It is supposed that as the pressure is reduced, so also is the initial compression of the medium, and so the device subsequently operates at a lower torque. The nature of the duration distribution at 40% load may indicate that the system is approaching the transition to subcritical behavior.

Furthermore, we have observed that some 100% load experiments possibly exhibit a Gaussian-type surplus of large events, akin to that of the supercritical state. The torque for these experiments is also higher than average. However, in these cases, the surplus is too small to be clearly identifiable, unlike the supercritical state, and the torque is still nearer the critical average than the supercritical average.

It is evident, therefore, that the critical state lies between the subcritical and supercritical states. Nonetheless, there still exists a regime of initial top plate loadings over which the resulting steady state torque can vary, *without altering the final state or probability distributions of the device*. In view of this result, we suggest that the critical state is not a universal attractor, but has a finite basin of attraction. Therefore, it is incorrect to assume that the SOC state observed is that proposed by BTW, which is understood to be an attracting state for *any* set of initial conditions.

CONCLUSION

Three modes of operation, or states have been identified in the dynamics of this granular stick-slip system. They correspond to the subcritical, critical, and supercritical states, also observed in earthquake phenomena [8]. The subcritical and supercritical states are highly repeatable and characterizable states, with easily identifiable features. The critical state exhibits a rich variety of behaviors, though its most important feature, the critical (i.e., power-law) exponents remain approximately constant for the size and energy. The duration distribution seems to be the most sensitive to the precise state of the medium, exhibiting two values with approximately equal likelihood, and would possibly be the first indication that an experiment may be moving towards a subcritical or a supercritical state.

The presence of scale invariance in the system has also been shown to be *reasonably* robust. That is, the regime of power-law behavior extends over a range of initial conditions, though this range does have boundaries.

Though we have previously concluded that SOC is the mechanism at work, the issue requires further exploration. The robustness of this state indicates that there is not merely an isolated critical point in the system's parameter space (as in, for example, the Ising model), but also a region about such a point for which a critical state is ultimately observed. Hence there is a *basin of attraction*, which is neither infinitesimally small nor infinitely big, but finite, being bounded by subcritical and supercritical states on either side. The original description of SOC proposed that the critical state should arise for virtually any set of system and initial conditions, clearly not the case here (nor, for that matter, in the well-known rice pile [13] or sandpaper-on-carpet [14] experiments).

However, any natural SOC system (assuming one exists) would intuitively also have boundaries to its basin of attraction. Indeed, Jensen states that ''self-organization to criticality will definitely occur only under certain conditions; one will always be able to generalize a model sufficiently to lose the critical behavior'' [15]. Thus there is a conceptual difference between ''self-organized criticality'' as defined by Bak *et al.* [1] and ''self-organization *to* criticality'' as here described by Jensen, the difference being essentially that of an *ideal* and a *real* SOC system. The ideal system is universally robust, but real systems have their limitations. In our apparatus the basin of attraction is externally selected, but within this basin the system still self-organizes to the critical state. We, therefore, suggest that ''bounded self-organized criticality'' might be a better description of the critical state observed in our apparatus, and possibly other real systems.

ACKNOWLEDGMENTS

This research was funded by the University of Limerick Foundation and Enterprise Ireland. We are grateful to Gerry Daly for devoting considerable time and effort to this project, and thank Ian Clancy for helpful discussions.

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